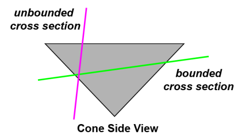
Supplementary Material for
MOCCA: Modeling and Optimizing Cone-joints for Complex Assemblies

The paper has mentioned supplementary material four times. We explain each of them following its order in the paper content.

1 Motion Cone Visualization



Choosing a proper cutting plane is needed to illustrate the 3D cone by its conic section. The inset shows a failure case where the conic section is an unbound region. Supposed that the plane has the form $\{\mathbf{x}|\mathbf{n}\cdot\mathbf{x}=1\}$, the normal $\mathbf{n}$ should be chosen to make the half-space $\{\mathbf{n}\cdot\mathbf{x}\geq 0\}$ contain the cones. More precisely, for cones in motion space, we choose the $\mathbf{n}$ to be the initial planar contact's normal appended with zeros. For cones in generalized normal space, we choose the $\mathbf{n}$ to be one of the part's translational disassembling directions appended with zeros.

2 Kinematic Based Infeasibility Measure

Computing the kinematic based infeasibility measure in Equation 1 is a dual problem of computing the forced-based infeasibility measure in Equation 2.

Dual Problem:

$$\begin{aligned} \max_{\boldsymbol{v}} \quad & \boldsymbol{w}^T\boldsymbol{v} - \frac{1}{2}\boldsymbol{v}^T\boldsymbol{v} \\ \text{s.t.} \quad & B_{\text{in}}\boldsymbol{v} \geq 0 \end{aligned} \tag{1}$$

Primal Problem:

$$\begin{aligned} \min_{\boldsymbol{f},\boldsymbol{s}} \quad & \frac{1}{2}\boldsymbol{s}^T\boldsymbol{s} \\ \text{s.t.} \quad & A_{\text{eq}}\boldsymbol{f} + \boldsymbol{s} = -\boldsymbol{w}, \\ & \boldsymbol{f} \geq 0 \end{aligned} \tag{2}$$

Proof:

The Lagrange function of Equation 2 is:

$$L(\boldsymbol{f},\boldsymbol{s},\boldsymbol{\mu},\boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{s}^T\boldsymbol{s} + \boldsymbol{\mu}^T(A_{\text{eq}}\boldsymbol{f} + \boldsymbol{s} + \boldsymbol{w}) - \boldsymbol{\lambda}^T\boldsymbol{f} \tag{3}$$

The dual Lagrange function $g(\boldsymbol{\mu},\boldsymbol{\lambda})$ is defined as the infimum of the Lagrange function $L(\boldsymbol{f},\boldsymbol{s},\boldsymbol{\mu},\boldsymbol{\lambda})$ in variables $\boldsymbol{f},\boldsymbol{s}$.

$$g(\boldsymbol{\mu},\boldsymbol{\lambda}) = \inf_{\boldsymbol{f},\boldsymbol{s}} L(\boldsymbol{f},\boldsymbol{s},\boldsymbol{\mu},\boldsymbol{\lambda}) \tag{4}$$

The Lagrange function L is a quadratic function which reaches its minimum when both its derivatives with respect to $\boldsymbol{f},\boldsymbol{s}$ are zero.

$$\frac{\partial L}{\partial \boldsymbol{f}} = A_{\text{eq}}^T\boldsymbol{\mu} - \boldsymbol{\lambda} = 0 \tag{5}$$

$$\frac{\partial L}{\partial \boldsymbol{s}} = \boldsymbol{s} + \boldsymbol{\mu} = 0 \tag{6}$$

Substitute Equation 5 and 6 into the Lagrange function (Equation 3).

$$g(\boldsymbol{\mu},\boldsymbol{\lambda}) = -\frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\mu} + \boldsymbol{\mu}^T\boldsymbol{w} \tag{7}$$

where the lagrange multipliers $\boldsymbol{\lambda}$ must be non-negative. Due to the strong duality theorem, the dual of the optimization problem (Equation 2) is:

$$\begin{aligned} \max_{\boldsymbol{\mu}} \quad & -\frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\mu} + \boldsymbol{\mu}^T\boldsymbol{w} \\ \text{s.t.} \quad & A_{\text{eq}}^T\boldsymbol{\mu} - \boldsymbol{\lambda} = 0, \\ & \boldsymbol{\lambda} \geq 0 \end{aligned} \tag{8}$$

Since $B_{\text{in}} = A_{\text{eq}}^T$, by renaming $\boldsymbol{\mu}$ to $\boldsymbol{v}$ in Equation 8, we prove that Equation 1 is a dual problem of Equation 2.

3 Infeasibility Derivatives $\frac{\partial E(\boldsymbol{w},\{\bar{\mathbf{V}}_{i,j}\})}{\partial \boldsymbol{\Psi}_{i,j}}$

This section explains how to compute the first order derivatives of infeasibility energy $E(\boldsymbol{w},\{\bar{\mathbf{V}}_{i,j}\})$ with respect to the approximated motion cones' variables $\boldsymbol{\Psi}_{i,j}$.

Our kinematic design uses an off-the-shelf interior point method to solve the optimization problem. Common solvers like BFGS require the first-order derivatives. The infeasibility energy $E(\mathbf{w},\{\bar{\mathbf{V}}_{i,j}\})$ is formulated as Equation 1, where the matrix B_{in} is a function of $\boldsymbol{\Psi}_{i,j}$ and the vector $\boldsymbol{w}$ is constant. Amos et al. [2019] proposed a sensitive analysis tool that computes the derivatives for series of quadratic programming problems including Equation 1. Their core idea is that any optimal solution of Equation 1 must satisfy the following KKT conditions.

$$\begin{aligned} \boldsymbol{\mu} &\geq 0 && \text{positive multipliers} \\ -\boldsymbol{\mu}^T(B_{\text{int}}\boldsymbol{v}) &= 0 && \text{complementary} \\ \boldsymbol{w} - \boldsymbol{v} - \boldsymbol{\mu}^T B_{\text{int}} &= 0 && \text{stationarity} \end{aligned} \tag{9}$$

where the $\boldsymbol{\mu}$ is the Lagrange multipliers of the constraints $B_{\text{int}}\boldsymbol{v} \geq 0$. Let's ignore the inequality constraints $\boldsymbol{\mu} \geq 0$ and take partial derivatives on both sides of the equalities in Equation 9:

$$\begin{aligned} \left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\Psi}_{i,j}}\right)^T B_{\text{int}}\boldsymbol{v}^* + \boldsymbol{\mu}^{*T} B_{\text{int}}\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\Psi}_{i,j}} &= -\boldsymbol{\mu}^{*T}\frac{\partial B_{\text{in}}}{\partial \boldsymbol{\Psi}_{i,j}}\boldsymbol{v}^* \\ \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\Psi}_{i,j}} + \left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\Psi}_{i,j}}\right)^T B_{\text{int}} &= -\boldsymbol{\mu}^{*T}\frac{\partial B_{\text{in}}}{\partial \boldsymbol{\Psi}_{i,j}} \end{aligned} \tag{10}$$

where $\boldsymbol{v}^*$ and $\boldsymbol{\mu}^*$ are the optimal solutions for the optimization problem. We can solve this large sparse linear system (Equation 10) to obtain the derivatives $\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\Psi}_{i,j}}$ and $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\Psi}_{i,j}}$. In practice, this system is pre-factorized to reduce solving time. The derivatives of infeasibility energy then are:

$$\frac{\partial E(\boldsymbol{w},\{\bar{\mathbf{V}}_{i,j}\})}{\partial \boldsymbol{\Psi}_{i,j}} = (\boldsymbol{w} - \boldsymbol{v}^*)^T\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\Psi}_{i,j}} \tag{11}$$

Moreover, we can adpot the same strategy to solve the optimization problem in geometric realization whose objective is also a quadratic function. Its derivatives can be computed by the same sensitive analysis approach.

4 New Interlocking Test

Supposed that all cone joints used in an assembly are not degenerated, we use a special external force $\boldsymbol{w}_{\text{int}} = -\sum_{i<j}(\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \boldsymbol{d}_{i,j}$ to verify the assembly's globally interlocking property. Here, $\boldsymbol{d}_{i,j}$ is chosen such that $(\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \boldsymbol{d}_{i,j} \geq 0$ and $(\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \boldsymbol{d}_{i,j} = 0$ if only if $\boldsymbol{v}_j = \boldsymbol{v}_i$. The assembly is globally interlocking if only if the infeasibility energy (Equation 1) under the external force $\boldsymbol{w}_{\text{int}}$ is zero.

Proof:

To avoid all parts in the assembly moving together, let's assume the key part and one of the remaining parts are fixed. The assembly is globally interlocking if no part can move in this circumstance.

For each cone joint whose motion space is $\mathbf{V}_{i,j}$, the associated two parts' velocities satisfy $\boldsymbol{v}_j - \boldsymbol{v}_i \in \mathbf{V}_{i,j}$. There always exists a constant vector $\boldsymbol{d}_{i,j}$, often chosen to be the initial planar contact normal appended with extra zeros, such that $(\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \boldsymbol{d}_{i,j} \geq 0$.

Moreover, if the cone $\mathbf{V}_{i,j}$ is not degenerated, the condition $(\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \boldsymbol{d}_{i,j} = 0$ holds if only if $\boldsymbol{v}_j = \boldsymbol{v}_i$. This constant vector $-\boldsymbol{d}_{i,j}$ acts like a repulsive force, which pushes the two adjacent parts away from each other. Since the external force $\boldsymbol{w}_{\text{int}}$ is set to be the summation of all repulsive forces $-\sum_{i,j}(\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \boldsymbol{d}_{i,j}$. The infeasibility energy is zero if only if each term $(\boldsymbol{v}_j - \boldsymbol{v}_i) \cdot \boldsymbol{d}_{i,j} = 0$. All parts must have the same velocity (i.e., zero) and therefore the assembly is globally interlocking.

Implementing this special external force $\boldsymbol{w}_{\text{int}}$ into our kinematic-geometric optimization framework allows designing new globally interlocking assemblies.

References

AMOS, B., AND KOLTER, J. Z., 2019. Optnet: Differentiable optimization as a layer in neural networks.